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Combinatorial PDEs on Cayley and coset graphs

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ABSTRACT

Building on the work by E. Barletta and S. Dragomir (2002) [3], this paper solves the initial value problems for the combinatorial heat and wave equations on Cayley and coset graphs.

1. Introduction and motivation

This work is motivated by the work of Barletta and Dragomir [3] where they deal with the combinatorial analogues of certain partial differential equations such as heat and wave equations. In particular, for a fixed positive integer N, they study the above mentioned partial differential equations on Hamming graphs whose vertex set is $\mathbb{Z}_2^N$. For other generalizations of [3] and related problems, see the works by Abatangelo and Dragomir [1,2].

A graph $G = (V(G), E(G))$ (in short, $G = (V, E)$ or G) consists of a set of vertices, denoted $V(G)$, and a set of edges, denoted by $E(G)$. All graphs in this paper are connected, finite and do not have loops or multiple edges. For any two vertices $x, y \in V$, the notation $x \sim y$ indicates $(x, y) \in E$ or equivalently x is adjacent to y in G. We write $m(x)$ to indicate the degree of a vertex $x \in V$.

Definition 1.1 (*Cayley Graph*)**.** Let Γ be a finite group with e as its identity element. Let S be a subset of Γ with $e \notin S$ such that S generates Γ and $S = S^{-1}$. The Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ on Γ with respect to S has $V(G) = \Gamma$ and $E(G) = \{(x, xs) : x \in \Gamma,\ s \in S\}$.

Remark 1.1. The Cayley graph $\mathrm{Cay}(\Gamma, S)$ defined in Definition 1.1 is a connected metric graph, where for each $x, y \in \Gamma, x \sim y$ if there exists $s \in S$ such that $x^{-1}y = s$ and the metric $d(x, y)$ is defined by $d(x, y) = \min |\{r : y = xs_{i_1}s_{i_2}\cdots s_{i_r},\ \text{where } s_{i_1}, s_{i_2}, \ldots, s_{i_r} \in S\}|$.

Definition 1.2 (*Coset Graph*)**.** Let Γ be a finite group, H a subgroup of Γ and $S \subset \Gamma$ such that $S \cap H = \emptyset, S^{-1} = S$ and $H \cup S$ generates Γ. Then the set of all distinct left cosets of H in Γ is the vertex set of the coset graph $G = \mathrm{Cos}(\Gamma, H, S)$ and $E(G) = \{(xH, xHs) : x \in \Gamma, s \in S\}$.

Remark 1.2. Let the coset graph $G = \mathrm{Cos}(\Gamma, H, S)$ be defined as in Definition 1.2. Then

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1. a coset graph is a Cayley graph if $H = \{e\}$.
2. for all $xH, yH \in V(G)$, $xH \sim yH$ whenever $x^{-1}y \in HSH$.
3. under the condition that H is a normal subgroup of Γ, the graph $\mathrm{Cos}(\Gamma, H, S)$ is a Cayley graph of the quotient group Γ/H with respect to $\{Hs : s \in S\}$.
4. it may happen that $Hs_i = Hs_j$ for some $s_i \neq s_j \in S$. In this case, the contribution of s_i and s_j to $E(G)$ remains the same. Therefore, from S, we extract $\widetilde{S}$ such that the elements of $\widetilde{S}$ give all the distinct right cosets of H in S.

For further results and references on Cayley and coset graphs, see [5]. Now, let $L^2(V)$ denote the set of complex valued functions on V. Then the *Combinatorial Laplacian* operator, $\Delta : L^2(V) \to L^2(V)$ on G is defined by

$$\Delta f(x) = m(x)f(x) - \sum_{y\sim x} f(y) \quad \text{for each } x \in V. \tag{1}$$

We end this subsection by stating the definition of the difference operator which acts as the differentiation operator in the discrete time analogue.

Definition 1.3. Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Then for any function $v : \mathbb{Z}_+ \to \mathbb{C}$, we define

$$\partial v(n) = v(n+1) - v(n).$$

The paper is arranged as follows. The character theory of finite groups is used to describe the Fourier transform on finite groups in Section 1.1. Using the results in Section 1.1 and building on the work in [3], we solve the initial value problems for the combinatorial heat equation and the combinatorial wave equation for the Cayley and coset graphs in Sections 2 and 3, respectively. In particular, our results imply that the paper solves the initial value problem for the combinatorial heat equation and the wave equation for all vertex transitive graphs.

1.1. Fourier transform for finite groups

Let Γ be a finite group, $\mathbb{C}^\times$ be the set of non-zero complex numbers and $\gamma : \Gamma \to \mathbb{C}^\times$ be a character of Γ. It is known that γ is a group homomorphism. The trivial character, denoted γ_0, maps each element of Γ to 1. We now state one of the well known results in character theory [4] (see pages 122–128) that is frequently used in this paper. We also define the Fourier transform and the inverse Fourier transform after this result. For related results and references on representations of finite groups and characters, see [4].

Lemma 1.3. *Let $\gamma_1, \ldots, \gamma_r$ be the complete set of inequivalent irreducible characters of a finite group Γ with dimensions $d_1, \ldots, d_r$, respectively. Then*

$$\frac{1}{|\Gamma|}\sum_{i=1}^{r} d_i\gamma_i(g) = \begin{cases} 0, & g \neq e, \\ 1, & g = e. \end{cases}$$

Definition 1.4. Let $f : \Gamma \to \mathbb{C}$ be a function on a finite group Γ. Then the Fourier transform of f, denoted $\mathfrak{F}(f)$ or $\hat{f}$, at γ is defined by

$$\mathfrak{F}(f)(\gamma) = \sum_{x\in\Gamma} \gamma(x)f(x).$$

Furthermore, if $\gamma_1, \ldots, \gamma_r$ is the complete set of inequivalent irreducible characters of Γ with dimensions $d_1, \ldots, d_r$, respectively, then the inverse Fourier transform of $\hat{f}$, denoted $\mathfrak{F}^{-1}(\hat{f})$, at x is defined by

$$\mathfrak{F}^{-1}(\hat{f})(x) = \frac{1}{|\Gamma|}\sum_{i=1}^{r} d_i\gamma_i(x^{-1})\hat{f}(\gamma_i).$$

Definition 1.5. Let $f, g : \Gamma \to \mathbb{C}$ be functions on a finite group Γ. The convolution map of f and g, denoted $f * g$, at x is defined by

$$f * g(x) = \sum_{y\in\Gamma} f(y)g(xy^{-1}).$$

We now state some relevant properties of Fourier transforms which are referred to in this paper. Even though these results can be easily derived from Definition 1.4 and Lemma 1.3, we give the proof for the sake of completeness.

Lemma 1.4. *Let $f, g : \Gamma \to \mathbb{C}$ be functions on a finite group Γ and let γ be a character of Γ. For any subset B of Γ, let $\chi_B(x)$ denote the characteristic function of B. Then*
1. $\mathfrak{F}(f * g)(\gamma) = \mathfrak{F}(f)(\gamma)\mathfrak{F}(g)(\gamma)$.
2. $f_y(x) = f(xy^{-1}) \Rightarrow \mathfrak{F}(f_y)(\gamma) = \gamma(y)\mathfrak{F}(f)(\gamma)$.
3. $\mathfrak{F}^{-1}(\mathbf{1})(x) = \chi_{\{e\}}(x)$, *where* $\mathbf{1}$ *is the constant function which takes the value* 1.

Proof. Let γ be any character of the group Γ. Then γ is a morphism and hence using Definition 1.4, one has

$$\begin{aligned}\mathfrak{F}(f*g)(\gamma) &= \sum_{x\in\Gamma}\gamma(x)\left(\sum_{y\in\Gamma}f(y)g(xy^{-1})\right) = \sum_{y\in\Gamma}f(y)\left(\sum_{x\in\Gamma}\gamma(x)g(xy^{-1})\right)\\ &= \sum_{y\in\Gamma}f(y)\sum_{z\in\Gamma}\gamma(zy)g(z) = \left(\sum_{y\in\Gamma}\gamma(y)f(y)\right)\left(\sum_{z\in\Gamma}\gamma(z)g(z)\right)\\ &= \mathfrak{F}(f)(\gamma)\mathfrak{F}(g)(\gamma).\end{aligned}$$

This completes the proof of the first part. For the second part, we similarly have

$$\begin{aligned}\mathfrak{F}(f_y)(\gamma) &= \sum_{x\in\Gamma}\gamma(x)f_y(x) = \sum_{x\in\Gamma}\gamma(x)f(xy^{-1}) = \sum_{z\in\Gamma}\gamma(zy)f(z) = \sum_{z\in\Gamma}\gamma(z)\gamma(y)f(z)\\ &= \gamma(y)\sum_{z\in\Gamma}\gamma(z)f(z) = \gamma(y)\mathfrak{F}(f)(\gamma).\end{aligned}$$

For the third part, let $\gamma_1,\ldots,\gamma_r$ be the complete set of inequivalent irreducible characters of Γ with respective dimensions $d_1,\ldots,d_r$. Then using Lemma 1.3, we have

$$\mathfrak{F}^{-1}(\mathbf{1})(x) = \frac{1}{|\Gamma|}\sum_{i=1}^{r}d_i\gamma_i(x^{-1})\mathbf{1}(\gamma_i) = \frac{1}{|\Gamma|}\sum_{i=1}^{r}d_i\gamma_i(x^{-1}) = \chi_{\{e\}}(x).$$

Hence, the proof of the lemma is complete. □

2. The combinatorial heat equation

We are now ready to solve the combinatorial analogue of the heat equation with discrete time variable on the Cayley and the coset graphs. The result for the Cayley graph is stated as follows.

Theorem 2.1. *Let S be a subset of a finite group Γ with $e\notin S$ such that S generates Γ and $S=S^{-1}$. Let G be the Cayley graph,* Cay(Γ,S)*, on Γ with respect to S. Then for any $f\in L^2(V)$, the combinatorial heat equation*

$$\Delta u(x,n) = \partial u(x,n) \quad \text{on } V\times\mathbb{Z}_+ \text{ with initial condition } u(x,0)=f(x) \tag{2}$$

admits a unique solution $u(x,n)=K_n*f(x)$, where

$$K_n(x) = \chi_{\{e\}}(x) + \sum_{j=1}^{n}(-1)^j\binom{n}{j}\underbrace{\mathfrak{F}^{-1}(a)*\cdots*\mathfrak{F}^{-1}(a)}_{j\text{ times}}(x)$$

with $\mathfrak{F}^{-1}(a)(x)=|S|\chi_{\{e\}}(x)-\chi_B(x)$ and $B\subset V$ is the boundary of the unit ball at e.

Proof. Let $S=\{s_1,\ldots,s_k\}$. Then G is a k-regular graph. Hence, using Eq. (1), one can rewrite Eq. (2) as

$$ku(x,n) - \sum_{i=1}^{k}u(xs_i,n) = u(x,n+1)-u(x,n) \quad\text{and}\quad u(x,0)=f(x).$$

Now taking the Fourier transform at a fixed character γ of Γ and using Lemma 1.4, we get

$$k\,\hat{u}(\gamma,n) - \left(\sum_{i=1}^{k}\gamma(s_i^{-1})\right)\hat{u}(\gamma,n) = \hat{u}(\gamma,n+1)-\hat{u}(\gamma,n);\quad \hat{u}(\gamma,0)=\hat{f}(\gamma).$$

Thus, $\hat{u}(\gamma,n+1)=[1+a(\gamma)]\hat{u}(\gamma,n)$, where $a(\gamma)=k-\sum_{i=1}^{k}\gamma(s_i^{-1})$ and $\hat{u}(\gamma,0)=\hat{f}(\gamma)$. Hence, an inductive argument gives

$$\hat{u}(\gamma,n) = [1+a(\gamma)]^n\hat{f}(\gamma). \tag{3}$$

Now, let us take the inverse Fourier transform of Eq. (3) and use Lemma 1.4 to get $u(x,n)=\mathfrak{F}^{-1}([1+a(.)]^n)*f(x)=K_n*f(x)$, where

$$K_n(x) = \mathfrak{F}^{-1}\left([1+a(.)]^n\right) = \sum_{j=0}^{n}\binom{n}{j}\mathfrak{F}^{-1}(a^j)(x) = \sum_{j=0}^{n}\binom{n}{j}\underbrace{\mathfrak{F}^{-1}(a)*\cdots*\mathfrak{F}^{-1}(a)}_{j\text{ times}}(x). \tag{4}$$

Thus, if we let $\gamma_1, \ldots, \gamma_r$ be the complete set of inequivalent irreducible characters of Γ with respective dimensions $d_1, \ldots, d_r$ then $\mathfrak{F}^{-1}(a)(x)$ in Eq. (4) is given by

$$\begin{aligned}\mathfrak{F}^{-1}(a)(x) &= \frac{1}{|\Gamma|}\sum_{i=1}^{r} d_i\gamma_i(x^{-1})a(\gamma_i) = k\mathfrak{F}^{-1}(\mathbf{1})(x) - \sum_{j=1}^{k}\frac{1}{|\Gamma|}\sum_{i=1}^{r} d_i\gamma_i(x^{-1})\gamma_i(s_j^{-1}) \\ &= k\chi_{\{e\}}(x) - \sum_{j=1}^{k}\frac{1}{|\Gamma|}\sum_{i=1}^{r} d_i\gamma_i((xs_j)^{-1}).\end{aligned} \tag{5}$$

As $\mathfrak{F}^{-1}(a)(e) = k$, using Lemma 1.3 and Eq. (5) for $x \neq e$, we get $\mathfrak{F}^{-1}(a)(x) \neq 0$ if and only if $x \in S$. Thus $\mathfrak{F}^{-1}(a)(x) = k\chi_{\{e\}}(x) - \chi_S(x) = k\chi_{\{e\}}(x) - \chi_B(x) = |S|\chi_{\{e\}}(x) - \chi_B(x)$. Since the Fourier inversion is unique, one gets the required unique solution to Eq. (2). □

Remark 2.2. Let m be a fixed positive integer and let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_m^N$, the Hamming distance $H(\mathbf{x}, \mathbf{y})$ is the number of coordinates where the vectors $\mathbf{x}$ and $\mathbf{y}$ disagree. The Hamming graph $\mathbf{H}$ has $V(\mathbf{H}) = \mathbb{Z}_m^N$ and $(\mathbf{x}, \mathbf{y}) \in E(\mathbf{H})$ if $H(\mathbf{x}, \mathbf{y}) = 1$. Hence, $\mathbf{H}$ is a d-regular graph with $d = (m-1)N$. If $\mathbf{e}_i \in \mathbb{Z}_m^N$ represents the vector that has 1 at the i-th place and zero elsewhere, then $\mathbf{H}$ arises as a Cayley graph of the group $\mathbb{Z}_m^N$ (addition modulo m) with $S = \{\ell\mathbf{e}_i : \ell = 1, \ldots, m-1,\ i = 1, \ldots, N\}$. Therefore, using Theorem 2.1, the solution to the heat equation on $\mathbf{H}$ can be easily obtained.

Let $G = \mathrm{Cos}(\Gamma, H, S)$ be a coset graph of Γ with respect to H and S. We now construct a new graph $\widetilde{G}$ with $V(\widetilde{G}) = \Gamma$. For $x, y \in \Gamma$, $x \sim y$ in $\widetilde{G}$ if there exist $b_iH, b_jH \in V(G)$ such that $x \in b_iH$, $y \in b_jH$, $b_iH \neq b_jH$ and $b_iH \sim b_jH$. That is, the induced subgraph of $\widetilde{G}$ with vertex set $b_iH \cup b_jH$ is a complete bipartite graph. Thus, $\widetilde{G}$ is a d-regular graph with $d = k|H|$, where $k = |\widetilde{S}|$ ($\widetilde{S}$ is defined as in Remark 1.2(4)). For any $f : V(G) \to \mathbb{C}$, we define $\widetilde{f} : V(\widetilde{G}) \to \mathbb{C}$ by $\widetilde{f}(x) = f(bH)$ whenever $x \in bH$. Thus, for a fixed left coset bH of H in Γ and $y, z \in bH$, $\widetilde{f}(y) = \widetilde{f}(z) = f(bH)$. We first therefore prove the following result.

Lemma 2.3. *Let H be a subgroup of a finite group Γ and $S \subset \Gamma$ such that $S \cap H = \emptyset$, $S^{-1} = S$ and $H \cup S$ generates Γ. Let G be the coset graph,* $\mathrm{Cos}(\Gamma, H, S)$, *of Γ with respect to H and S. Then, for the function $\widetilde{f}$ defined above, the combinatorial heat equation*

$$\frac{1}{|H|}\Delta_{\widetilde{G}}\widetilde{u}(x, n) = \partial\widetilde{u}(x, n) \quad \text{on } V(\widetilde{G}) \times \mathbb{Z}_+ \text{ with initial condition } \widetilde{u}(x, 0) = \widetilde{f}(x) \tag{6}$$

admits a unique solution and $\widetilde{u}(x, n)$ is a constant on each left coset of H.

Proof. Let $\{Hs_1, \ldots, Hs_k\}$ be the set of all distinct right cosets of H, where $s_i \in S$ for $1 \le i \le k$. By using Eq. (1), Eq. (6) reduces to

$$\frac{1}{|H|}\left(k|H|\widetilde{u}(x, n) - \sum_{h\in H}\sum_{i=1}^{k}\widetilde{u}(xhs_i, n)\right) = \widetilde{u}(x, n+1) - \widetilde{u}(x, n); \quad \widetilde{u}(x, 0) = \widetilde{f}(x). \tag{7}$$

Using a similar argument as in Theorem 2.1 (to obtain Eq. (3)), we get

$$\hat{\widetilde{u}}(\gamma, n) = [1 + \widetilde{a}(\gamma)]^n\hat{\widetilde{f}}(\gamma), \quad \text{where } \widetilde{a}(\gamma) = k - \frac{1}{|H|}\sum_{h\in H}\sum_{i=1}^{k}\gamma((hs_i)^{-1}). \tag{8}$$

Taking the inverse Fourier transform in Eq. (8) and by use of calculations similar to those used to get Eq. (4), we have

$$\widetilde{u}(x, n) = \widetilde{K}_n * \widetilde{f}(x), \quad \text{where } \widetilde{K}_n(x) = \sum_{j=0}^{n}\binom{n}{j}\underbrace{\mathfrak{F}^{-1}(\widetilde{a}) * \cdots * \mathfrak{F}^{-1}(\widetilde{a})}_{j \text{ times}}(x). \tag{9}$$

Since $\mathfrak{F}^{-1}(\widetilde{a})(e) = k$, using Lemma 1.3 for $x \neq e$, $\mathfrak{F}^{-1}(\widetilde{a})(x) \neq 0$ if and only if $x = (hs_i)^{-1}$ for all $h \in H$ and for all i, $1 \le i \le k$. Hence, $\mathfrak{F}^{-1}(\widetilde{a})(x) = k\chi_{\{e\}}(x) - \frac{1}{|H|}\sum_{i=1}^{k}\chi_{s_iH}(x)$. Since $S = S^{-1}$, $\{s_1H, \ldots, s_kH\} = \{s_1^{-1}H, \ldots, s_k^{-1}H\}$ and whenever $x = bh \in bH$, one has

$$\mathfrak{F}^{-1}(\widetilde{a}) * \widetilde{f}(x) = \sum_{y\in\Gamma}\mathfrak{F}^{-1}(\widetilde{a})(y)\widetilde{f}(xy^{-1}) = k\widetilde{f}(x) - \frac{1}{|H|}\sum_{i=1}^{k}\sum_{h\in H}\widetilde{f}(bhs_i^{-1}). \tag{10}$$

Thus, using Eq. (10), for any $x, z \in bH$, $\mathfrak{F}^{-1}(\widetilde{a}) * \widetilde{f}(x) = \mathfrak{F}^{-1}(\widetilde{a}) * \widetilde{f}(z)$. Hence, from Eqs. (9) and (10), we get the required solution to Eq. (6). □

We now state and prove the combinatorial heat equation for coset graphs. Since there is a one-to-one correspondence between coset graphs and vertex transitive graphs (see [5, Theorems 3.7, 3.8]), Theorem 2.4 can be used to get a solution of the combinatorial heat equation for any vertex transitive graph.

Theorem 2.4. *Let H be a subgroup of a finite group Γ and $S \subset \Gamma$ such that $S \cap H = \emptyset$, $S^{-1} = S$ and $H \cup S$ generates Γ. Let G be the coset graph,* $\mathrm{Cos}(\Gamma, H, S)$, *of Γ with respect to H and S. Then for any $f \in L^2(V(G))$, the combinatorial heat equation*

$$\Delta u(X,n) = \partial u(X,n) \quad \text{on } V(G) \times \mathbb{Z}_+ \text{ with initial condition } u(X,0) = f(X) \tag{11}$$

admits a unique solution if and only if the initial value problem represented by Eq. (6) admits a unique solution.

Proof. By Lemma 2.3, Eq. (6) admits a unique solution $\widetilde{u}(x,n)$, where $\widetilde{u}(x,n)$ is a constant on each left coset of H. Hence, Eq. (7) can be rewritten as

$$\frac{1}{|H|}\left(k|H|\widetilde{u}(x,n) - |H|\sum_{i=1}^{k}\widetilde{u}(xhs_i,n)\right) = \widetilde{u}(x,n+1) - \widetilde{u}(x,n);\ \widetilde{u}(x,0) = \widetilde{f}(x). \tag{12}$$

Let us define $u : V(G) \times \mathbb{Z}_+ \to \mathbb{C}$ by $u(bH,n) = \widetilde{u}(x,n)$ for each $x \in bH$. Then Eq. (12) can be rewritten in the form

$$k\,u(X,n) - \sum_{i=1}^{k} u(Xs_i,n) = u(X,n+1) - u(X,n); \quad u(X,0) = f(X), \tag{13}$$

where $X = bH$. Since, $\widetilde{u}(x,n)$ is a unique solution to Eq. (6), we get a unique solution to Eq. (11) from Eq. (13).

Similarly, if $u(X,n)$ is a solution to Eq. (11), one can define $\widetilde{u} : \widetilde{V(G)} \times \mathbb{Z}_+ \to \mathbb{C}$ by $\widetilde{u}(x,n) = u(bH,n)$ for all $x \in bH$. It can now be seen that $\widetilde{u}(x,n)$ is a solution to Eq. (6) and hence the uniqueness follows from Lemma 2.3. □

3. The combinatorial wave equation

In this section, we solve the combinatorial analogue of the wave equation with discrete time variable on Cayley and coset graphs. The result on Cayley graphs is stated next.

Theorem 3.1. *Let S be a subset of a finite group Γ with $e \notin S$ such that S generates Γ and $S = S^{-1}$. Let G be the Cayley graph,* $\mathrm{Cay}(\Gamma, S)$, *on Γ with respect to S and let $f, g \in L^2(V)$. Then, the combinatorial wave equation*

$$\Delta u(x,n) = \partial^2 u(x,n) \quad \text{on } V \times \mathbb{Z}_+ \text{ with initial conditions } u(x,0) = f(x),\ \partial u(x,0) = g(x) \tag{14}$$

has a solution if and only if $\widehat{g}(\gamma_0) = 0$ (γ_0 is the trivial character of Γ). Moreover, in this case, this solution is unique and is given by

$$\begin{aligned} u(x,n) &= f(x) + ng(x) + \sum_{i=1}^{[\frac{n}{2}]}(-1)^i\binom{n}{2i}\underbrace{\mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a)}_{i \text{ times}} * f(x) \\ &\quad + \sum_{i=1}^{[\frac{n}{2}]}(-1)^i\binom{n-1}{2i+1}\underbrace{\mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a)}_{i \text{ times}} * g(x), \end{aligned}$$

where $\mathfrak{F}^{-1}(a)(x) = |S|\chi_{\{e\}}(x) - \chi_B(x)$ and $B \subset V$ is the boundary of the unit ball at e.

Proof. Let $S = \{s_1, \ldots, s_k\}$. Then G is a k-regular graph. Therefore, taking the Fourier transform of Eq. (14), one gets the following difference equation.

$$\hat{u}(\gamma,n+2) - 2\hat{u}(\gamma,n+1) - \left[\left(k - \sum_{i=1}^{k}\gamma(s_i^{-1})\right) - 1\right]\hat{u}(\gamma,n) = 0,\ \hat{u}(\gamma,0) = \hat{f}(\gamma),\ \hat{u}(\gamma,1) - \hat{u}(\gamma,0) = \hat{g}(\gamma). \tag{15}$$

A solution of Eq. (15) is $\hat{u}(\gamma,n) = \lambda(1+\sqrt{a})^n + \mu(1-\sqrt{a})^n$, where $a(\gamma) = k - \sum_{i=1}^{k}\gamma(s_i^{-1})$ and λ, μ can be determined from the initial conditions. The initial conditions lead to linear equations $\lambda + \mu = \hat{f}(\gamma)$ and $\lambda(1+\sqrt{a}) + \mu(1-\sqrt{a}) = \hat{f}(\gamma) + \hat{g}(\gamma)$. Solving for the unknowns give $\lambda = \frac{\hat{f}(\gamma)}{2} + \frac{\hat{g}(\gamma)}{2\sqrt{a(\gamma)}}$ and $\mu = \frac{\hat{f}(\gamma)}{2} - \frac{\hat{g}(\gamma)}{2\sqrt{a(\gamma)}}$. These solutions are compatible if and only if $\hat{g}(\gamma_0) = 0$. But when $\hat{g}(\gamma_0) = 0$, one gets $\hat{u}(\gamma_0,n) = \hat{f}(\gamma_0)$ for the trivial representation and for any other representation $\gamma \neq \gamma_0$, one has

$$\hat{u}(\gamma,n) = \left[\sum_{i=0}^{[\frac{n}{2}]}\binom{n}{2i}a^i(\gamma)\right]\hat{f}(\gamma) + \left[\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2i+1}a^i(\gamma)\right]\hat{g}(\gamma).$$

Taking the inverse Fourier transform, we get $u(x, n) = F_n * f(x) + G_n * g(x)$, where

$$F_n(x) = \chi_{\{e\}}(x) + \sum_{i=1}^{[\frac{n}{2}]} (-1)^i \binom{n}{2i} \underbrace{\mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a)}_{i \text{ times}}(x),$$

$$\text{and} \quad G_n(x) = n\chi_{\{e\}}(x) + \sum_{i=1}^{[\frac{n-1}{2}]} (-1)^i \binom{n}{2i+1} \underbrace{\mathfrak{F}^{-1}(a) * \cdots * \mathfrak{F}^{-1}(a)}_{i \text{ times}}(x).$$

Thus, by Theorem 2.1, $\mathfrak{F}^{-1}(a)(x) = |S|\chi_{\{e\}}(x) - \chi_B(x)$ and the required result follows. □

Now, it can be observed that using Theorem 3.1 and Remark 2.2, the solution to the wave equation on a Hamming graph can be calculated. We are now ready to state and prove the combinatorial wave equation for coset graphs.

Theorem 3.2. *Let H be a subgroup of a finite group Γ and $S \subset \Gamma$ such that $S \cap H = \emptyset$, $S^{-1} = S$ and $H \cup S$ generates Γ. Let G be the coset graph, $\mathrm{Cos}(\Gamma, H, S)$, of Γ with respect to H and S. Then for any two given elements $f, g \in L^2(V(G))$ the combinatorial wave equation*

$$\Delta_G u(X, n) = \partial^2 u(X, n) \quad \text{on } V(G) \times \mathbb{Z}_+ \text{ with initial conditions } u(X, 0) = f(X),\ \partial u(X, 0) = g(X) \tag{16}$$

has a solution if and only if $\widehat{\widetilde{g}}(\gamma_0) = 0$, where γ_0 is the trivial character of the group Γ and $\widetilde{g} : \Gamma \to \mathbb{C}$ satisfies $\widetilde{g}(x) = g(bH)$ whenever $x \in bH$.

Proof. Construct graph $\widetilde{G}$ as in Lemma 2.3 and Theorem 2.4. Then Eq. (16) reduces to

$$\frac{1}{|H|}\Delta_{\widetilde{G}}\widetilde{u}(x, n) = \partial^2 \widetilde{u}(x, n) \quad \text{on } V(\widetilde{G}) \times \mathbb{Z}_+ \tag{17}$$

with initial conditions $\widetilde{u}(x, 0) = \widetilde{f}(x)$ and $\partial\widetilde{u}(x, 0) = \widetilde{g}(x)$, where $\widetilde{f}, \widetilde{g} : V(\widetilde{G}) \to \mathbb{C}$ satisfying $\widetilde{f}(x) = f(bH)$ and $\widetilde{g}(x) = g(bH)$ for all $x \in bH$.

With similar calculations as in Lemma 2.3, Theorems 2.4 and 3.1, the solution to Eq. (17) exists if and only if $\widehat{\widetilde{g}}(\gamma_0) = 0$. Hence Eq. (16) has a unique solution if and only if $\widehat{\widetilde{g}}(\gamma_0) = 0$. □

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